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Separating the regular and irregular energy levels and their statistics in a Hamiltonian system with mixed classical dynamics

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Abstract. We look at the high-lying eigenstates (from the 10 001 to the 13 000) in the Robnik billiard (defined as a quadratic conformal map of the unit disk) with the shape parameter $\lambda = 0.15$. All the 3000 eigenstates have been numerically calculated and examined in the configuration space and in the phase space which—in comparison with the classical phase space—enabled a clear cut classification of energy levels into regular and irregular. This is the first successful separation of energy levels based on purely dynamical rather than special geometrical symmetry properties. We calculate the fractional measure of regular levels as $\rho_1 = 0.365 \pm 0.01$, which is in remarkable agreement with the classical estimate $\rho_1 = 0.360 \pm 0.001$. This finding confirms the Percival's (1973) classification scheme, the assumption in Berry–Robnik (1984) theory and the rigorous result by Lazutkin (1981, 1991). The regular levels obey the Poissonian statistics quite well, whereas the irregular sequence exhibits the fractional power-law level repulsion and globally Brody-like statistics with $\beta = 0.286 \pm 0.001$. This is due to the strong localization of irregular eigenstates in the classically chaotic regions. Therefore, in the entire spectrum we see that the Berry–Robnik regime is not yet fully established so that the level spacing distribution is correctly captured by the Berry–Robnik–Brody distribution.

1. Introduction

In the early days of quantum chaos Percival (1973) suggested, using the semiclassical picture and the correspondence principle, that the eigenstates of a classically non-integrable and (partially) chaotic Hamiltonian system should be classified as regular or irregular, depending on whether they are associated with classically regular or chaotic regions in the phase space. In fact, he referred mainly to the sensitivity of the energy levels with respect to some family parameter (of a one-parameter family of Hamiltonians), by saying that in irregular levels the second differences (now known and studied as curvature) are typically much larger than in regular levels. This picture has been recently confirmed in the studies of curvature distribution of quantum spectra of classically non-integrable systems (Gaspard *et al* 1990, Takami and Hasegawa 1992) where it has been finally demonstrated that here again we encounter universality classes (Zakrzewski and Delande 1993, von Oppen 1994a, b).

On the other hand, the irregular levels have been assumed to be associated also geometrically with the classically chaotic regions in the sense of the principle of uniform semiclassical condensation (PUSC, see e.g. Li and Robnik 1994a, Berry 1977a, b), which

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states that a quantal phase-space distribution function like Wigner or Husimi, or anything in between in the semiclassical limit $\hbar \to 0$, uniformly condenses on the associated classical invariant object, which in this case is the underlying chaotic component. According to the same principle, the regular levels are associated with the quantized invariant tori. Thus, in the semiclassical limit the entire spectrum and the associated eigenstates are decomposed in the regular sequence and, generally, in the many irregular sequences. More quantatively, it is intuitively obvious to assume that the fractional level density of each level sequence is equal to the fractional phase-space volume of the underlying invariant classical component, and this is one of the main assumptions of the Berry–Robnik (1984) theory. It is also a rigorous result by Lazutkin (1981, 1991) that the torus quantization in the plane classical convex billiards works only for states where the classical invariant tori exist and the relative measure of such regular states, according to him, is equal to the fractional phase-space volume of the classically regular regions.

So far an inspection and separation of spectra in this sense has been performed only in one special case (Bohigas *et al* 1990a, b, 1993), where the regular levels are characterized by being almost degenerate pairs due to some discrete geometrical symmetry. It is the goal of our present paper to perform a separation of regular and irregular eigenstates on purely dynamical grounds in a generic system of mixed-type classical dynamics described by a KAM-type scenario, which is, to the best of our knowledge, the first such analysis. The system that we shall analyse numerically is the Robnik billiard (1983, 1984) (defined as a quadratic conformal map of the unit disc) with the shape parameter $\lambda = 0.15$, and this work is a natural continuation of our previous work (Li and Robnik 1995).

We have calculated the 3000 eigenstates between the 10 001 and the 13 000 of even parity in configuration space and in the phase space, and performed the classification of states in comparison with the classical dynamics (Poincaré surface of section plots). After this separation of regular and irregular states, we have performed a complete spectral statistical analysis which we present in section 3. As we shall see the ingredients of the Berry–Robnik picture are almost completely confirmed, except for the localization phenomena in the chaotic states which are responsible for the deviation from GOE statistics to which ideally the statistics must converge in the strict semiclassical limit $\hbar \to 0$. These phenomena have recently been demonstrated, discussed and partially (qualitatively) explained by Prosen and Robnik (1994b).

2. The preliminaries: the calculational technique and the method of analysis

The theoretical questions that we addressed in the introduction are of course completely general, but in order to illustrate them we have to confine ourselves to some specific and possibly generic system. For such a purpose it is ideal to study billiard systems. We have chosen the Robnik billiard (defined as a quadratic conformal map of the unit disk $w = z + \lambda z^2$) with the shape parameter $\lambda = 0.15$, which is a convex plane billiard. This system and the conformal mapping diagonalization technique have been introduced by Robnik (1983, 1984) and further studied by Berry and Robnik (1986), Robnik and Berry (1986), Hayli *et al* (1987), Frisk (1990), Bruus and Stone (1994), Stone and Bruus (1993a, b), Markarian (1993), Prosen and Robnik (1993a, b, 1994b), Li and Robnik (1994a, 1995) and Bäcker *et al* (1994). This one-parameter system (called the Robnik billiard or Robnik model by most workers in the field) is a nice generic system which at $\lambda = 0$ is the integrable circle billiard, for $0 < \lambda \leqslant 1/4$ it is a typical KAM system (the KAM theorem applies because the boundary is analytic and convex), for $\lambda > 1/4$ it becomes non-convex and eventually for some sufficiently large $\lambda \leqslant 1/2$ becomes ergodic, mixing and K-system.

Markarian (1993) has recently rigorously proved this property for $\lambda = 1/2$ (the cardiod billiard). Li and Robnik (1994b) have numerical evidence that ergodicity (and mixing and the K property) already exists at $\lambda \geqslant 0.2775$, but could set in even earlier. More details about the system and the numerical technique are given in our recent papers (Li and Robnik 1994a, 1995).

In calculating the eigenfunctions in configuration space we have used Heller's method (1991) of plane wave decomposition by applying the singular value decomposition (Press *et al* 1986). All the technical details and tricks have been described in Li and Robnik (1994a, 1995). The problem with missing eigenstates has been overcome by using the exact (double precision: 16 digits) eigenenergies for all 3000 consecutive states, from the 10 001 to the 13 000 state. The exact energy levels have been obtained by the conformal mapping diagonalization technique described in detail in Prosen and Robnik (1993a).

We have calculated the eigenfunctions in configuration space, their Wigner functions in phase space, also their projections onto the surface of the section, and finally also their Husimi type smoothed objects on the surface of the section. All the definitions, formulae and the method of presentation are exactly as introduced in Prosen and Robnik (1993b), and later employed in Li and Robnik (1995). Therefore, and because we shall not show any plots of eigenfunctions in this paper, we refer the reader to those previous works. The quantal phase-space plots (smoothed projections of Wigner functions onto SOS) have been compared with the classical SOS plots for all the 3000 states. (Of course we have also plotted the configurational eigenfunctions, but this is not so important for the classification of states.) One smaller, but representative, part of this output (200 eigenstates) will be published separately (Li and Robnik 1994c) and is also available from the authors upon request. Some representative eigenstates are shown in Li and Robnik (1995).

In spite of the vast amount of our material (altogether 2×3000 plots) there was no difficulty in classifying the eigenstates. A state is declared regular if it is localized on an invariant torus belonging to the classical regular region in the classical SOS. A state is declared irregular if it is localized (either uniformly or non-uniformly) on a classically chaotic region in the SOS. According to the PUSC we expect that in the strict semiclassical limit $\hbar \to 0$, or equivalently when the energy goes to infinity, the irregular states become uniformly extended on the classical chaotic component. However, before this limit is reached we observe strong localization of irregular states due to the slow classical diffusion inside the underlying classical chaotic component. In fact the vast majority of our irregular states are localized chaotic states and only a few irregular states are extended chaotic. The consequences of such localization for the spectral statistics have been recently discussed and demonstrated by Prosen and Robnik (1994b). For the associated irregular level sequence we find the phenomenon of the fractional power-law level repulsion and globally a Brody-like behaviour which then approaches GOE deeper in the semiclassical limit of smaller and smaller effective $\hbar$. This is confirmed again in our present paper in section 3.

In performing the classification of states we had 109 cases, which at first sight would be classified as mixed states in the sense that they 'live' both partially in classically regular regions and partially in irregular regions. However a closer examination of the finer details of the quantal phase-space functions (by decreasing the smoothing length) led us to the conclusion that they should be classified as regular states. Other *a posteriori* reasons for such a decision will be described in the next section.

It is true that our separation procedure by eye might be considered slightly subjective but *a posteriori* it has proved successful, confirming the general opinion that human eye is a good judge of structures and patterns. Another reason for our elementary approach is the fact that a machine method represents a major numerical difficulty. Nevertheless, progress in this direction has been lately reported in an application of a suitable method of calculating the classical-quantum overlap integrals in Prosen (1995c).

The central result of this paper is the counting of regular and irregular states. Among 3000 states we found 987 regular, 109 mixed and 1904 irregular states. The percentages are 32.9%, 3.6% and 63.5%, respectively. This is the result of the preliminary classification. With the decision mentioned before and further explained in section 3 we absorb the mixed states into the set of regular states (so now there are $987 + 109 = 1096$ of them), which then results in $\rho_1 = 0.365 \pm 0.01$ for the regular states and $\rho_2 = 1 - \rho_1 = 0.635 \pm 0.01$ for irregular states. This is in excellent agreement with the classical relative phase-space volume of the regular regions as calculated and reported by Prosen and Robnik (1993a) where $\rho_1 = 0.360 \pm 0.001$. We believe that this is quite striking confirmation not only of Percival's scheme (1973), but also of the main assumption in the Berry–Robnik (1984) theory of energy level spacings and more specifically of Lazutkin's rigorous results (1981, 1991) on convex plane billiards.

In order to get some idea about the convergence of these numerical figures with the size of the sample we should note that examination of an additional 1000 consecutive states, namely from the 13 001 to the 14 000 state, and the classification by the same criteria as above gives the result $\rho_1 = 0.360 \pm 0.01$, which is then in brilliant agreement with the classical value. However, due to the statistical fluctuations all the statistical spectral measures do not necessarily improve monotonically as we shall comment in the next section.

We believe that this is the first detailed dynamical analysis of a large sample of eigenstates enabling the separation of regular and irregular states based on the classification procedure explained above. This successful separation of regular and irregular states makes it possible to analyse the statistical properties of regular and irregular sequences separately with the goal to confirm the aspects of the Berry–Robnik approach (1984) which is the subject of the next section.

3. Statistics of energy level sequences

The subject of the Berry–Robnik (1984) theory is the statistics of energy spectra of quantal Hamiltonians whose classical counterparts have mixed classical dynamics in the sense that classical regular regions and classical irregular regions coexist in the phase space. It is based on our knowledge of spectral fluctuations and their statistics in the context of quantum chaos (Berry 1983, Giannoni *et al* 1991, Haake 1991, Gutzwiller 1990, Eckhardt 1988, Bohigas 1991, Robnik 1994). We know that there are three universality classes of spectral fluctuations: Poisson statistics in the classically integrable cases; in the case of classical ergodicity we find the GOE/GUE statistics of random matrix theories depending on whether there is one/none anti-unitary symmetry (we ignore spin). The interesting and difficult case of mixed-type classical dynamics of KAM-like (generic) systems was studied for the first time by Robnik (1984) numerically, where a continuous transition from Poisson to GOE statistics in a billiard system (Robnik 1983) was found, and this work has been substantially revised in Prosen and Robnik (1993a). Further theoretical progress was published by Berry and Robnik (1984) where the following semiclassical theory of the level spacings has been presented. The eigenstates (their Wigner functions in phase space) are supposed to condense uniformly on the underlying classical invariant regions such that each of them—in the semiclassical limit—supports a level sequence which for itself has Poisson or GOE statistics if the region is regular or irregular, respectively. All the regular regions can be thought of as supporting a single Poisson sequence because the Poisson statistics are preserved upon a statistically independent superposition. The mean level spacing of such a sequence is determined by the fractional phase-space volume of the regular regions. On the other hand each chaotic (GOE) level sequence has a mean level spacing governed by the corresponding fractional phase-space volume. The entire spectrum is then assumed to be a statistically independent superposition of all subsequences. The statistical independence in the semiclassical limit is justified by the principle of uniform semiclassical condensation of eigenstates (in the phase space) and by the lack of their mutual overlap, in consistency with Percival's (1973) conjecture. Thus the problem of the statistics of the entire spectrum is now mathematically precisely formulated (this forms the essence of the Berry–Robnik approach) and its solution, as far as the level spacings are concerned, can be expressed in the following way. The statistical independence of superposition implies factorization of the gap distribution functions (Mehta 1991, Haake 1991). The probability that there is no level within a gap clearly factorizes upon a statistically independent superposition. The connection between the level spacing distribution $P(S)$ and the gap distribution $E(S)$ is as follows

$$P(S) = \frac{\mathrm{d}^2 E(S)}{\mathrm{d}S^2} \tag{1}$$

and conversely

$$E(S) = \int_S^\infty \mathrm{d}x\,(x - S)P(x). \tag{2}$$

Leaving aside the general case of infinitely many chaotic components which does not include anything surprisingly new let us restrict to the case of one regular component with mean level density ρ_1 (= fractional phase-space volume) and one chaotic component with the mean level density ρ_2 where $\rho_1 + \rho_2 = 1$. This is already going to be an excellent approximation because in a generic system of a mixed type there is usually only one large and dominating chaotic region. Following Mehta (1991), Haake (1991) and Berry and Robnik (1984) we have

$$E(S) = E_{\mathrm{Poisson}}(\rho_1 S)E_{\mathrm{GOE}}(\rho_2 S) \tag{3}$$

where the Poissonian gap distribution E_{Poisson} is

$$E_{\mathrm{Poisson}}(S) = \exp(-S) \tag{4}$$

whereas for the E_{GOE} there is no simple closed formula (for the infinitely dimensional GOE case) and it must be worked out by using practical approximations for P_{GOE} and/or E_{GOE} which, for example, can be found in Haake (1991, pp 72–4). However, the two-dimensional GOE case (the so-called Wigner surmise) can be worked out explicitly as given in Berry and Robnik (1984, formula (28)), which is usually a good starting approximation.

As for the delta statistics $\Delta(L)$ a similar procedure based on the assumption of statistical independence leads to the simple (additive) formula (Seligman and Verbaarschot 1985)

$$\Delta(L) = \Delta_{\mathrm{Poisson}}(\rho_1 L) + \Delta_{\mathrm{GOE}}(\rho_2 L) \tag{5}$$

where $\Delta_{\mathrm{Poisson}}(L) = L/15$, while for Δ_{GOE} there are good approximations given in Bohigas (1991).

The main objective of this paper is to verify the elements and aspects of the Berry–Robnik approach. However, before such a regime is formed in a KAM system in the ultimate far semiclassical limit we typically observe a quasi-universal behaviour in the spectral statistics which is characterized by the fractional power-law level repulsion and globally the adequacy of the Brody (1973, Brody *et al* 1981) distribution and of similar distributions such as that of Izrailev (1989). A thorough numerical study of this phenomenon has been published recently by Prosen and Robnik (1993a, 1994a, b), and will be discussed later on. In cases where the chaotic component becomes large ($\rho_2 \approx 1$) and the quantal chaotic states strongly localized, the phenomenological Berry–Robnik–Brody distribution proposed in Prosen and Robnik (1994b) can be very efficient in capturing the global features of the experimental and numerical data (Lopac *et al* 1992, 1994).

As explained in section 2 we have separated 1096 regular levels from the remaining 1904 irregular levels. Now we perform the statistical analysis of each subsequence in the next two subsections 3.1–3.2 and then also of the entire spectrum in subsection 3.3.

3.1. Regular levels

Of course, before performing the separation procedure we have carefully unfolded the entire spectrum of 3000 levels by using the Weyl formula (with perimeter and curvature corrections) for even-parity states given in formula (4) of our paper (Li and Robnik 1994a). After extracting 1096 regular levels and calculating the (nearest neighbour) level spacings we have first to normalize the empirical level spacing distribution $P(S)$ to the unit first moment, i.e. such that the mean level spacing is unity. By doing this we again obtain the relative level density of the regular component with the value of $\rho_1 = 0.365$. Having performed this normalization we calculate the cumulative level spacing distribution $W(S) = \int_0^S P(x)\,dx$ and compare it with the Poisson statistics and also try to fit it with the best-fit Brody distribution

$$W^B(S,\beta) = 1 - \exp(-bS^{\beta+1}) \qquad b = \{\Gamma((\beta+2)/(\beta+1))\}^{\beta+1} \tag{6}$$

and best-fit Berry–Robnik distribution (3). The result is shown in figure 1, where we also show the so-called U-function introduced in Prosen and Robnik (1993a), defined as

$$U(W) = \frac{2}{\pi} \arccos\sqrt{1-W} \tag{7}$$

in reference with the best-fit Berry–Robnik distribution. As we see the agreement of our numerical data with the Poissonian distribution is reasonably good in the sense that Poisson is within the $\pm 1\sigma$ statistical fluctuations of the data. The best-fit Brody distribution and the best-fit Berry–Robnik distribution deviate slightly from Poisson: for Brody we get $\beta = 0.041 \pm 0.002$, which ideally should be zero, and for Berry–Robnik we get $\rho_1 = 0.704$, which ideally should be unity. In figure 2 we show the delta statistics for the same sequence of 1096 regular levels and for sufficiently small $L < L_{max}$ (for L larger than L_{max} the saturation effects set in Berry (1985): in our case $L_{max} \approx 10$) we see excellent agreement with Poisson. The best-fit Seligman–Verbaarschot delta statistics (5) gives $\rho_1 = 0.9998$, which is surprisingly close to the ideal value unity. Thus we think that our statistical analysis of the regular sequence of levels clearly supports the theoretical assumptions in the Berry–Robnik (1984) approach.

3.2. Irregular levels

In analogy to the previous procedure we have extracted 1904 irregular levels and normalized the level spacing distribution to the unit first moment which yields $\rho_2 = 0.634 \pm 0.002$ in consistency with $\rho_1 = 0.365 \pm 0.002$, such that we have unit level density of the entire spectrum, namely $\rho_1 + \rho_2 = 1$. The cumulative level spacing distribution for the irregular sequence is plotted in figure 3, where we see that it deviates from the ideal GOE distribution substantially and significantly. This is a consequence of the strong localization of irregular eigenstates on classically chaotic components demonstrated for even higher energies (ten

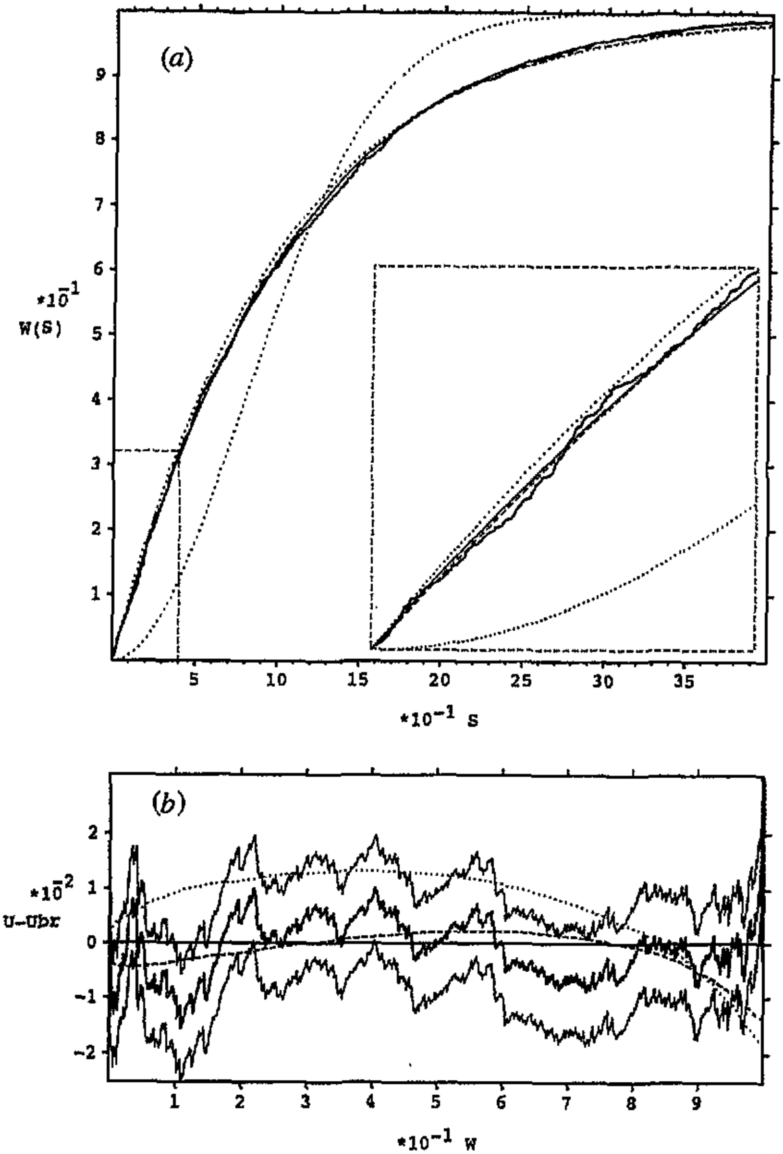

Figure 1. The cumulative level spacing distribution (*a*) and *U*-function (*b*) of 1096 regular levels. In (*a*) the dotted curves represent the Poisson and GOE statistics. The thick curve is the numerical data, while the thin curve shows the best-fit Berry–Robnik distribution and the broken curve shows the best-fit Brody distribution. The best-fit parameters are $\rho_1 = 0.704$ and $\beta = 0.041 \pm 0.002$, respectively. In (*b*) we display the *U*-function difference $U(W) - U(W^{BR})$ against $W(S)$, where $U_{br} = U(W^{BR}(S))$ is the *U*-function of the best-fit Berry–Robnik distribution. The numerical noisy curves (the average value with $\pm 1\sigma$ band) in (*b*) are compared with the best-fit Brody distribution (broken curve) and the Poisson distribution (dotted curve).

times larger) in Li and Robnik (1995) for the same billiard with $\lambda = 0.15$, whereas the implications for level statistics have been demonstrated and discussed by Prosen and Robnik (1994b). The best-fit Berry–Robnik distribution yields $\rho_1 = 0.374$, which, however, ideally

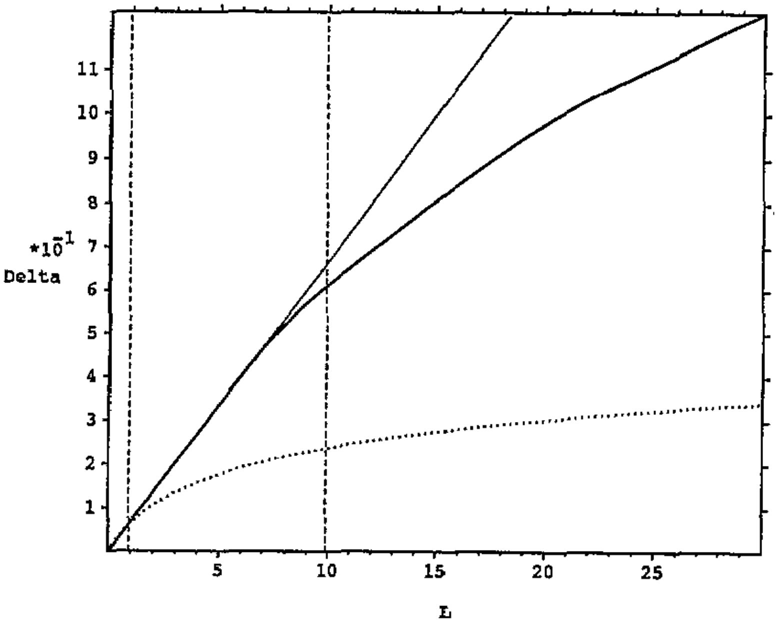

Figure 2. The delta statistics of 1096 regular levels and the best-fit Seligman–Verbaarschot formula (5). The dotted curves are the Poisson and GOE, the thick curve shows the numerical data and the best-fit Seligman–Verbaarschot formula is represented by the thin curve which overlaps the limiting Poissonian at $L < 7.5$. The vertical dashed lines indicate the region where the least-squares fit has been performed. The best-fit parameter value is $\rho_1 = 0.9998$ which is in excellent agreement with the ideal value 1.

should be zero. On the other hand, the more interesting best-fit Brody distribution captures the fractional power-law level repulsion with $\beta = 0.286 \pm 0.001$, and as we see both in the $W(S)$ as well as in the U-function plot it is globally a much better fit than Berry–Robnik, which is qualitatively well understood in Prosen and Robnik (1994b). If we were able to go to much higher energies, say 100 times higher, then the statistics of irregular levels are predicted to approach GOE statistics, i.e. the Brody parameter β goes to one.

As for the delta statistics the analysis of the data by the best-fit Seligman–Verbaarschot formula (5) gives $\rho_1 = 0.373$, which is equal to and consistent with the Berry–Robnik fit for level spacings, but of course is at variance with the ideal value $\rho_1 = 0$. This is certainly the consequence of the localization effects. Having understood the reasons for the deviation of spectral statistics from the limiting (as $\hbar \to 0$ or equivalently as the energy goes to infinity) GOE statistics we may conclude that the relevant aspect of the Berry–Robnik (1984) theory is reconfirmed, supporting our claim that this theory is asymptotically exact theory.

3.3. The entire spectrum

Ideally in the strict semiclassical limit of sufficiently high energies we would expect the statistics of the entire spectrum of 3000 levels (regular plus irregular) to be described by the Berry–Robnik distribution (3). This prediction is based on two facts: (i) GOE applies for irregular levels, and (ii) regular and irregular levels are superposed statistically independently. Now, the first assumption is not satisfied due to the localization effects explained in the preceding subsection, so that instead of GOE we find in fact Brody with

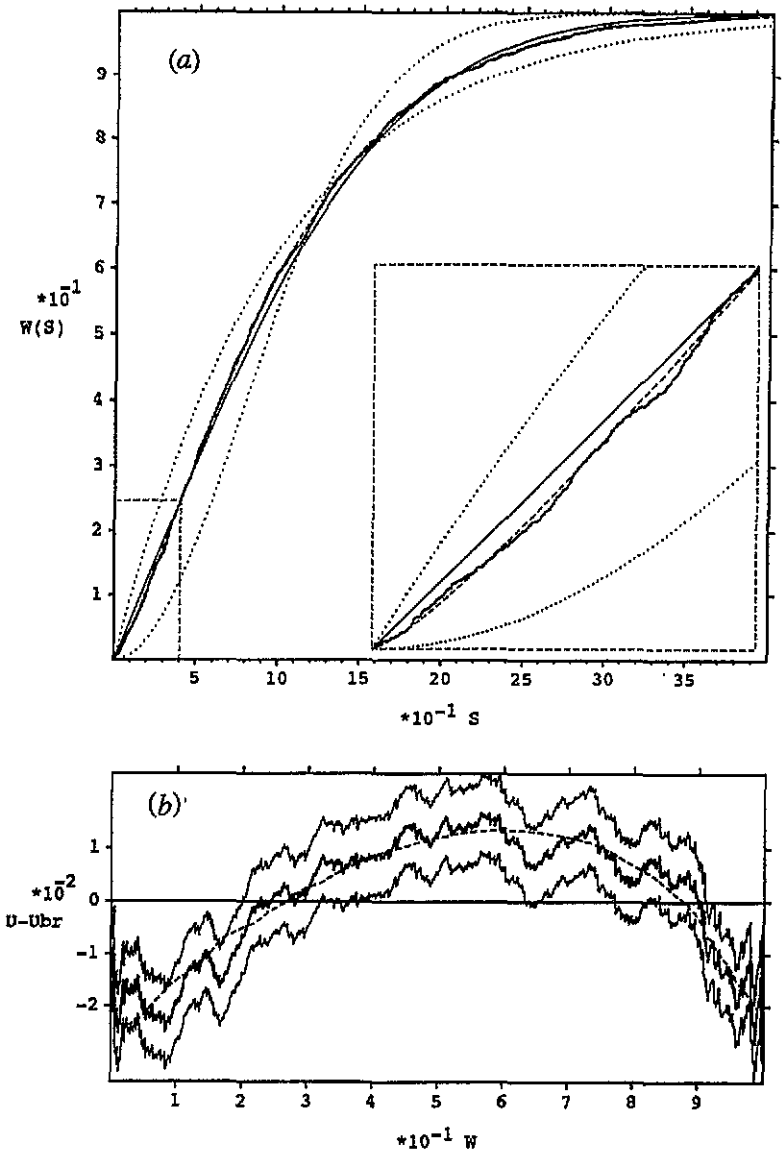

Figure 3. The same as figure 1 but for the 1904 irregular levels. The best-fit parameter values are $\rho_1 = 0.374$ and $\beta = 0.286 \pm 0.001$ for the Berry–Robnik and Brody distributions, respectively. In the U-function plot (b) we see that the Brody fit is globally very good. (We do not show the Poissonian curve here.)

$\beta = 0.286 \pm 0.001$, as shown in figure 3. The second assumption of statistical independence is accepted as valid and thus the reasoning leading to the factorization of the gap distribution is valid except that we now find not the Berry–Robnik distribution but the so-called Berry–Robnik–Brody (Prosen and Robnik 1994b) distribution instead

$$E^{\mathrm{BRB}}(S, \rho_1, \beta) = E_{\mathrm{Poisson}}(\rho_1 S) E^{\mathrm{B}}(\rho_2 S, \beta) = \mathrm{e}^{-\rho_1 S} E^{\mathrm{B}}((1 - \rho_1) S, \beta) \qquad (8)$$

where Brody statistics have a gap distribution which can be expressed in terms of incomplete Gamma function Q

$$E^{\mathrm{B}}(S,\beta) = Q\left(\frac{1}{\beta+1}, \left(\Gamma(\frac{\beta+2}{\beta+1})S\right)^{\beta+1}\right). \tag{9}$$

Indeed this phenomenological two-parameter distribution provides an excellent fit to our spectral data of 3000 consecutive levels as can be seen in figure 5, with the best-fit parameter values $\rho_1 = 0.309$ and $\beta = 0.370$. They differ slightly from the ideal values $\rho_1 = 0.365$ and $\beta = 0.286 \pm 0.001$, but are consistent with them, thereby confirming the statistical independence assumption.

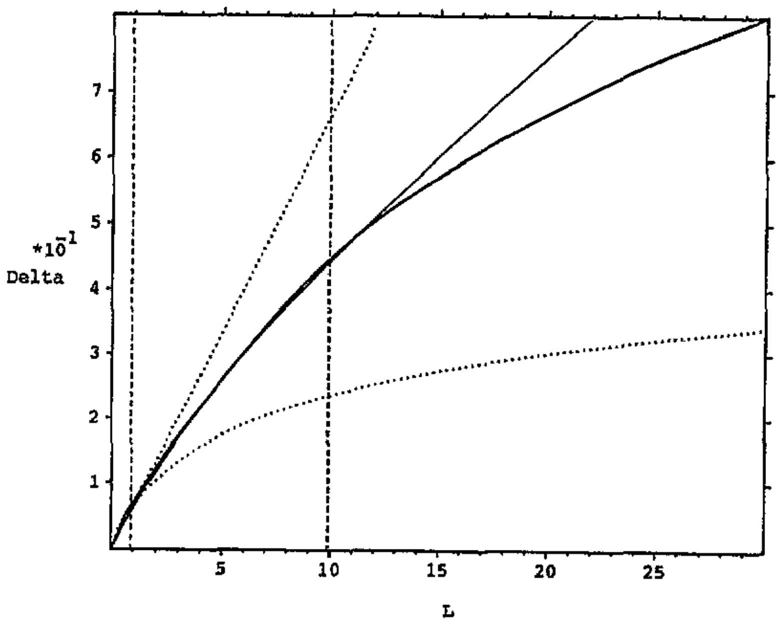

Figure 4. The same as figure 2 but for the 1904 irregular levels. The best-fit parameter value is $\rho_1 = 0.373$ which is at variance with the ideal value 0, but is completely consistent with the Berry–Robnik fit in figure 3.

Nevertheless, just as a consistency check we can try to fit the spectrum with the Brody distribution and the Berry–Robnik distribution. The result is shown in figure 6 and the best-fit parameters are $\beta = 0.166 \pm 0.007$ and $\rho_1 = 0.504$, respectively. The fits are not bad and Brody is certainly better than Berry–Robnik as can be seen especially in the U-function plot. The conclusion is that as a consequence of the localization effects the entire spectrum is indeed best described by the Berry–Robnik–Brody distribution fitted in figure 5. The value of χ^2 in the latter is about three times smaller than in Brody and six times smaller than in Berry–Robnik.

Finally we should mention that the best-fit Seligman–Verbaarschot (5) delta statistics fitted to the spectrum within the interval $0.9 < L < 9.9$, below the saturation region, give $\rho_1 = 0.472$, which is consistent with the Berry–Robnik fit with $\rho_1 = 0.504$ as explained above. For the sake of completeness we show the delta statistics in figure 7.

Unlike the counting measure of regular levels the statistical measures of the spectral subsequences do not converge so uniformly and so fast. Calculating the level spacing

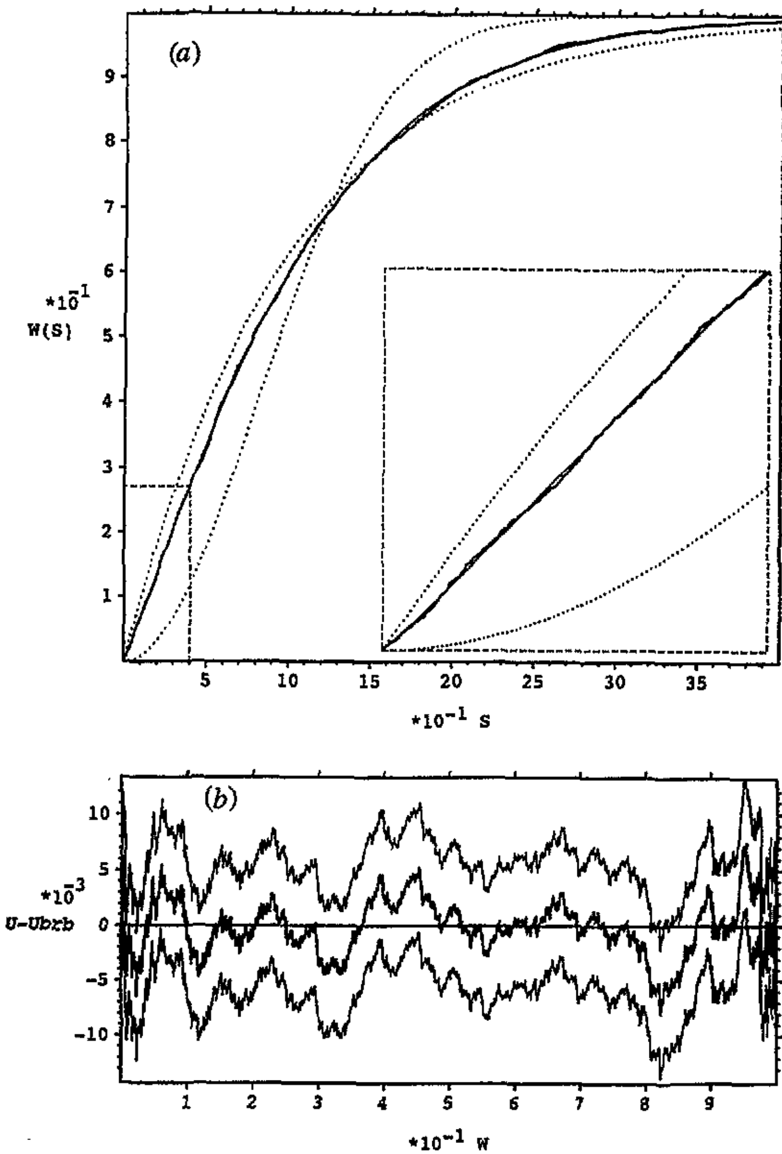

Figure 5. The cumulative level spacing distribution (a) and U-function plot (b) of the total 3000 regular and irregular levels compared with the best-fit Berry–Robnik–Brody formula (8). In (a) the dotted curves represent the Poisson and GOE statistics. The bold curve is the numerical data, while the thin curve shows the best-fit Berry–Robnik–Brody distribution. The best-fit parameters are $\rho_1 = 0.309$ and $\beta = 0.370$. In (b) we show the U-function difference $U(W) - U(W^{\mathrm{BRB}})$ against $W(S)$, where $U_{\mathrm{brb}} = U(W^{\mathrm{BRB}}(S))$ is the U-function of the Berry–Robnik–Brody. The bold curve represents numerical data and the noisy curves indicate the $\pm 1\sigma$ band. The quality of the fit is really excellent. However, the ideal values of ρ_1 and β would be $\rho_1 = 0.360$ and, according to figure 3, $\beta = 0.286$.

distribution and the delta statistics of an enlarged sample (namely 1000 more levels added on top to the block of 3000 levels) led actually to slightly worse results.

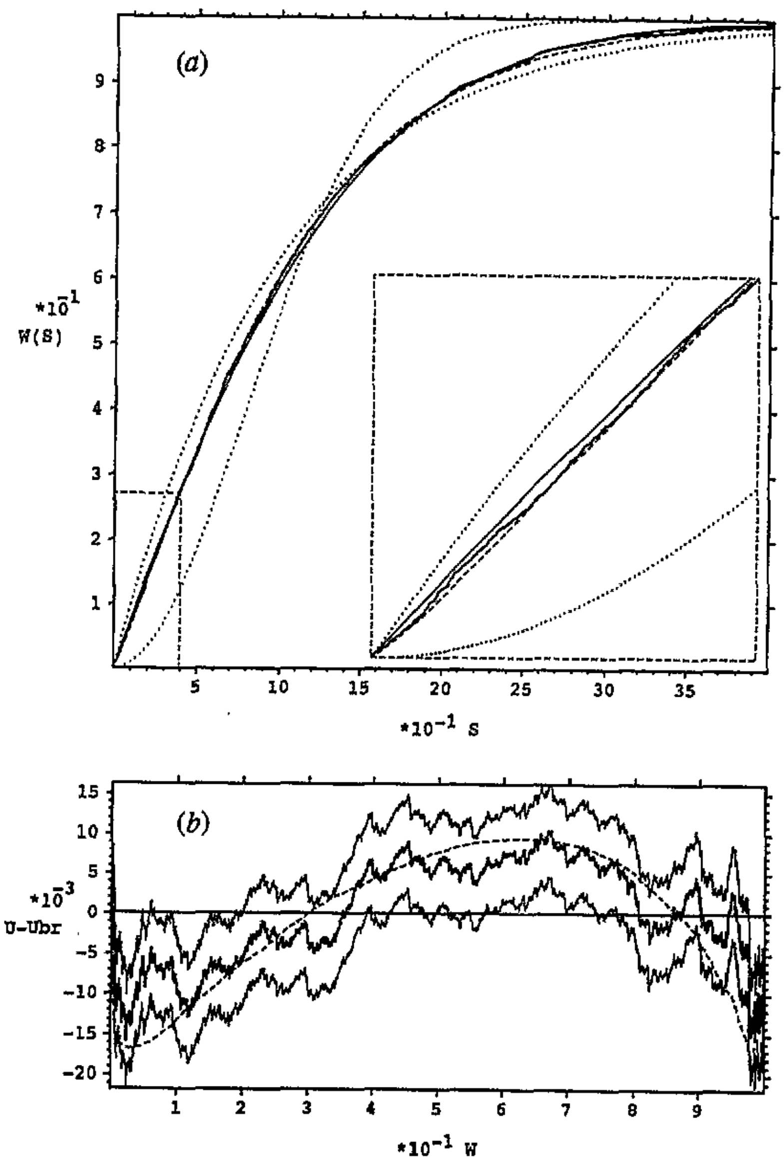

Figure 6. The same as figure 3 but for the total 3000 levels. The best-fit parameter values are $\rho_1 = 0.504$ and $\beta = 0.166 \pm 0.007$.

Finally, as promised, we would like to give additional *a posteriori* reasons for classifying the preliminary mixed-type states as regular. (i) Due to the strong localization any ambiguous mixed state is likely at least to mimic a regular state. (ii) After absorbing the mixed states into the set of regular ones the relative fraction of regular states $\rho_1 = 0.365$ agrees much better with the classical estimate $\rho_1 = 0.360$. (iii) The statistical measures ($P(S)$ and $\Delta(L)$) then also agree much better with Poisson statistics. (iv) The statistical measures of irregular sequence then agree notably better with the trend towards GOE in the sense that the Brody exponent β increases from $\beta = 0.276$ to $\beta = 0.286$.

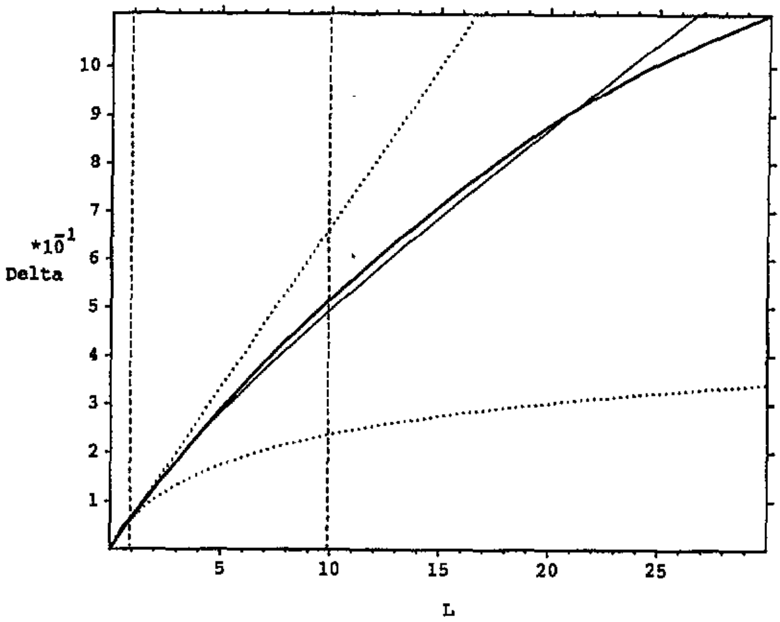

Figure 7. The same as figure 2 but for the total 3000 levels. The best-fit parameter value is $\rho_1 = 0.472$, which is in complete agreement with the Berry–Robnik fit in figure 6.

4. Discussions and conclusions

The main goal of our present paper is to analyse a large sample of eigenstates of an autonomous Hamiltonian system sufficiently high in the semiclassical limit, and to perform the classification of the states into regular and irregular by examining their Husimi-type phase-space plots in correspondence with the classical surface of section plots. This plan has materialized in the specific choice of the Robnik billiard with the shape parameter $\lambda = 0.15$, which is a KAM-type system (the boundary is convex and analytic) with one large and dominating chaotic component with the relative phase-space volume (not SOS area) $\rho_2 = 0.640$. We have plotted 3000 consecutive states (from the 10 001 to the 13 000) in configuration and in phase space, and performed the said classification with the result that the relative fraction of regular states is equal to 0.365 which is in excellent agreement with the classical value $\rho_1 = 0.360 = 1 - \rho_2$. Furthermore we have calculated the statistical measures (level spacing distribution and the delta statistics) for the regular and irregular sequence separately and also of the entire spectrum. The aspects and assumptions of the Berry–Robnik (1984) approach have been confirmed, although we are not yet far enough in the semiclassical limit to see the GOE statistics for the irregular levels, while the regular levels are seen to obey the Poisson statistics very well. We believe that this is the first direct dynamical separation of regular and irregular levels in a generic system and the first such statistical analysis. It confirms the Percival's (1973) classification scheme, the ingredients in the Berry–Robnik theory (1984) and the specific rigorous results on convex billiards by Lazutkin (1981,1991). It supports our claim that the Berry–Robnik (1984) theory is an asymptotically exact theory. One of the future projects should be in going further into the semiclassical limit for which new methods are needed, and one such might be the employment of the quantum Poincaré mapping (Bogomolny 1992, Schanz and Smilansky

1994, Prosen 1994a, b, 1995a–c). This might lead us to a direct demonstration of the applicability of the Berry–Robnik theory with high numerical and statistical significance, which so far was successful only in an abstract time-dependent Hamiltonian system, namely the compact standard map (Prosen and Robnik 1994a, b).

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References

Bäcker A, Steiner F and Stifter P 1994 *Preprint* DESY 94-213
Berry M V 1977a *Phil. Trans. R. Soc.* **287** 237
——1977b *J. Phys. A: Math. Gen.* **10** 2083
——1983 *Chaotic Behaviour of Deterministic Systems (Proc. NATO ASI Les Houches Summer School)* ed G Iooss, R H G Helleman and R Stora (Amsterdam: Elsevier) p 171
——1985 *Proc. R. Soc.* A **400** 229
——1989 *Proc. R. Soc.* A **423** 219
Berry M V and Robnik M 1984 *J. Phys. A: Math. Gen.* **17** 2413
——1986 *J. Phys. A: Math. Gen.* **19** 649
Bohigas O 1991 *Chaos and Quantum Systems (Proc. NATO ASI Les Houches Summer School)* ed M-J Giannoni, A Voros and J Zinn-Justin (Amsterdam: Elsevier) p 87
Bohigas O, Tomsovic S and Ullmo D 1990a *Phys. Rev. Lett.* **64** 1479
——1990b *Phys. Rev. Lett.* **65** 5
——1993 *Phys. Rep.* **223** 4
Bogomolny E B 1992 *Nonlinearity* **5** 805
Brody T A 1973 *Lett. Nuovo Cimento* **7** 482
Brody T A, Flores J, French J B, Mello P A, Pandey A and Wong S S M 1981 *Rev. Mod. Phys.* **53** 385
Bruus H and Stone A D 1994 *Phys. Rev.* B **50** 18275
Eckhardt B 1988 *Phys. Rep.* **163** 205
Frisk H 1990 Nordita *Preprint*
Gaspard P, Rice S A, Mikeska H J and Nakamura K 1990 *Phys. Rev.* A **42** 4015
Giannoni M-J, Voros J and Zinn-Justin J (ed) 1991 *Chaos and Quantum Systems* (Amsterdam: North-Holland)
Gutzwiller M C 1990 *Chaos in Classical and Quantum Mechanics* (New York: Springer)
Haake F 1991 *Quantum Signatures of Chaos* (Berlin: Springer)
Hayli A, Dumont T, Moulin-Ollagier J and Strelcyn J M 1987 *J. Phys. A: Math. Gen.* **20** 3237
Heller E J 1991 *Chaos and Quantum Systems (Proc. NATO ASI Les Houches Summer School)* ed M-J Giannoni, A Voros and J Zinn-Justin (Amsterdam: Elsevier) p 547
Izrailev F M 1989 *J. Phys. A: Math. Gen.* **22** 865
Lazutkin V F 1981 *The Convex Billiard and the Eigenfunctions of the Laplace Operator* (Leningrad: University Press) (in Russian)
Lazutkin V F 1991 *KAM Theory and Semiclassical Approximations to Eigenfunctions* (Heidelberg: Springer)
Li Baowen and Robnik M 1994a *J. Phys. A: Math. Gen.* **27** 5509
——1994b *J. Phys. A: Math. Gen.* to be submitted
——1994c *Preprint* CAMTP/94-11 to be submitted
——1995 *J. Phys. A: Math. Gen.* **28** 2799
Lopac V, Brant S and Paar V 1992 *Phys. Rev.* A **45** 3503
——1994 submitted
Markarian R 1993 *Nonlinearity* **6** 819
Mehta M L 1991 *Random Matrices* (San Diego: Academic)
Percival I C 1973 *J. Phys. B: At. Mol. Phys.* **6** L229

Press W H, Flannery B P, Teukolsky S A and Vetterling W T 1986 *Numerical Recipes* (Cambridge: Cambridge University Press)
Prosen T and Robnik M 1993a *J. Phys. A: Math. Gen.* **26** 2371
——1993b *J. Phys. A: Math. Gen.* **26** 5365
——1994a *J. Phys. A: Math. Gen.* **27** L459
——1994b *J. Phys. A: Math. Gen.* **27** 8059
Prosen T 1994a *J. Phys. A: Math. Gen.* **27** L709
——1994b *Preprint* CAMTP/94-3 to be published in *Open Systems & Information Dynamics* 1996
——1995a *Preprint* CAMTP/94-9 *J. Phys. A: Math. Gen.* **28** in press
——1995b *Preprint* CAMTP/95-1 *J. Phys. A: Math. Gen.* **28** in press
——1995c *Preprint* CAMTP/95-2 *Physica* D submitted
Robnik M 1983 *J. Phys. A: Math. Gen.* **16** 3971
——1984 *J. Phys. A: Math. Gen.* **17** 1049
——1994 *J. Phys. Soc. Japan Suppl.* **63** 131
Robnik M and Berry M V 1986 *J. Phys. A: Math. Gen.* **19** 669
Schanz H and Smilansky U 1993 *Preprint* Weizmann Institute
Seligman T H and Verbaarschot J J M 1985 *J. Phys. A: Math. Gen.* **18** 2227
Stone A D and Bruus H 1993a *Physica* B **189** 43
——1993b *Surf. Sci.* **305** 490
Takami T and Hasegawa H 1992 *Phys. Rev. Lett.* **68** 419
von Oppen F 1994a *Phys. Rev. Lett.* **73** 798
——1994b *Phys. Rev. Lett.* to appear (private communication from H A Weidenmüller)
Zakrzewski J and Delande D 1993 *Phys. Rev.* E **47** 1650